SOLUTION OF ALGEBRAIC EQUATIONS IN INFINITE SERIES.

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(*Read April 25, 1908.*)

I. Introduction.

1. The object of this investigation is to develop a method for determining all the roots, real and imaginary, of an algebraic equation by means of infinite series.

2. Suppose the given equation to be represented by $f(y)=0$. The method consists in introducing a factor x into all the terms but two of the given equation; expanding y, which now is an algebraic function of x, into a power series in x; placing x equal to unity in this power series. The resulting value of y, if convergent, is a root of the given equation expressed in terms of the coefficients and exponents of the equation.

3. The method presupposes the solution of the two-term equation

$$ay^n+b=0.$$

In fact the roots of this equation when written in the form

$$y^n=-\frac{b}{a}=r(\cos\theta+i\sin\theta)$$

are found to any required degree of approximation from the formula

$$y=r^{\frac{1}{n}}\left(\cos\frac{2s\pi+\theta}{n}+i\sin\frac{2s\pi+\theta}{n}\right),$$

where

$$s=0, 1, 2, 3, 4, \cdots, n-1.$$

4. The method proceeds step by step from the two-term equation to the three-term equation, from the three-term equation to the four-term equation, and so on.

II. The Three-term Equation.

5. In the three-term equation

$$ay^n + by^k + c = 0$$

the two terms from which the x is to be omitted can be selected in three different ways. This gives rise to the three equations

$$(1) \qquad ay^n + by^k x + c = 0$$

$$(2) \qquad ay^n + by^k + cx = 0$$

$$(3) \qquad ay^n x + by^k + c = 0$$

each one of which defines y as an algebraic function of x.

6. Values of y expressed as power series in x may be found from each one of these three equations by any one of the following three methods, which, however, are essentially the same.

7. *The Multinomial Theorem.*—Assume that the power series for y is

$$(4) \qquad y = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + \cdots.$$

The multinomial theorem asserts that the coefficient of x^r in the expansion of y^n is

$$(5) \qquad \Sigma \frac{n!}{q_0!\, q_1!\, q_2! \cdots q_s!} p_0^{q_0} p_1^{q_1} p_2^{q_2} \cdots p_s^{q_s} x^{q_1+2q_1+3q_3+\cdots sq_s},$$

provided

$$(6) \qquad q_1 + 2q_2 + 3q_3 + \cdots sq_s = r$$

$$(7) \qquad q_0 + q_1 + q_2 + \cdots q_s = n.$$

The expansion of y^k is obtained in like manner.

Assuming that the power series (4) represents the algebraic function defined by equation (1), the substitution of the expansions of y^n and y^k in equation (1) must give an identity. This identity is

$$(8) \qquad 0 = \left.\begin{array}{l} ap_0^n \\ +c \end{array}\right| \left.\begin{array}{l} +anp_0^{n-1}p_1 \\ +bp_0^k \end{array}\right| x + \left.\begin{array}{l} \dfrac{an(n-1)}{1 \cdot 2} p_0^{n-2}p_1^2 \\ +anp_0^{n-1}p_2 \\ +bkp_0^{k-1}p_1 \end{array}\right| x^2 + \left.\begin{array}{l} \dfrac{an(n-1)(n-2)}{1 \cdot 2 \cdot 3} p_0^{n-3}p_1^3 \\ +an(n-1)p_0^{n-2}p_1p_2 \\ +anp_0^{n-1}p_3 \\ +\dfrac{bk(k-1)}{1 \cdot 2} p_0^{k-2}p_1^2 \\ +bkp_0^{k-1}p_2 \end{array}\right| x^3 + \cdots$$

In this identity the coefficient of each power of x equals zero. Hence p_0 is the root of the two-term equation

$$ap_0 + c = 0.$$

The coefficient of the first power of x equated to zero determines p_1 uniquely in terms of p_0; the coefficient of x^2 equated to zero determines p_2 uniquely in terms of p_0 and p_1; in general, the coefficient of x^s equated to zero determines p_s uniquely in terms of $p_0, p_1, p_2, \cdots, p_{s-1}$. All the successive coefficients of the power series (4) are therefore determined uniquely in terms of p_0, any one of the roots of the two-term equation $ap_0^n + c = 0$.

The power series representing the algebraic functions defined by equations (2) and (3) are determined in precisely the same manner. Unfortunately if the coefficients of the power series are determined in this way it is difficult to recognize the law which will enable one to write the general term of the power series, which is necessary for the application of a convergency test.

When x is made unity, the equations (1), (2) and (3) become the three-term equation

$$ay^n + by^k + c = 0$$

and the power series, if convergent when $x = 1$, becomes the solution of this equation.

If it is known in advance that some one of equations (1), (2), (3) furnishes a power series which is convergent when $x = 1$, the multinomial theorem determines in an elementary and direct manner the coefficients of the power series.

8. *Maclaurin's Series.*—The algebraic function y defined by the equation

$$ay^n + by^k x + c = 0$$

can be expanded into a power series in x by means of Maclaurin's series

$$(9) \qquad y = y_0 + \frac{dy_0}{dx_0}x + \frac{d^2y_0}{dx_0^2}\frac{x^2}{1 \cdot 2} + \frac{d^3y_0}{dx_0^3}\frac{x^3}{1 \cdot 2 \cdot 3} + \cdots.$$

The expansion is identical in form with the expansion obtained by means of the multinomial theorem and consequently has the same disadvantage.

9. *Lagrange's Theorem.*—The equation

$$ay^n + by^k x + c = 0$$

may be written

$$(10) \qquad y^n = -\frac{c}{a} - x\frac{b}{a}y^k.$$

Placing $y^n = z$, whence $y = z^{1/n}$, this equation becomes

$$(11) \qquad z = -\frac{c}{a} - x\frac{b}{a}z^{\frac{k}{n}}.$$

Lagrange's theorem asserts that if

$$z = v + x\phi(z)$$

$$(12) \qquad \begin{aligned} f(z) = f(v) + x\phi(v)f'(v) + \frac{x^2}{1\cdot 2}\frac{d}{dv}\{\overline{\phi(v)}^2 f'(v)\} + \cdots \\ + \frac{x^n}{n!}\frac{d^{n-1}}{dv^{n-1}}\{\overline{\phi(v)}^n f'(v)\}\cdots. \end{aligned}$$

If now

$$f(z) = z^{\frac{1}{n}}, \quad \phi(z) = -\frac{b}{a}z^{\frac{k}{n}},$$

and after the derivatives in series (12) have been formed v is replaced by $-c/a$, there results, making x unity,

$$(13) \qquad \begin{aligned} z^{\frac{1}{n}} = y = \left(-\frac{c}{a}\right)^{\frac{1}{n}} + \frac{b}{nc}\left(-\frac{c}{a}\right)^{\frac{1+k}{n}} + \frac{b^2}{2!\,n^2c^2}(1+2k-n)\left(-\frac{c}{a}\right)^{\frac{1+2k}{n}} \\ + \frac{b^3}{3!\,n^3c^3}(1+3k-n)(1+3k-2n)\left(-\frac{c}{a}\right)^{\frac{1+3k}{n}} \\ + \frac{b^4}{4!\,n^4c^4}(1+4k-n)(1+4k-2n)(1+4k-3n)\left(-\frac{c}{a}\right)^{\frac{1+4k}{n}} + \cdots. \end{aligned}$$

In series (13) the law of formation of the successive terms is evident and this law is readily proved by induction by using Lagrange's theorem.

Series (13) may be more concisely written by placing

$$\left(-\frac{c}{a}\right)^{\frac{1}{n}} = y_0,$$

so that y_0 is a root of the two-term equation

$$ay_0^n + c = 0,$$

and denoting the continued product

$$(1 + sk - n)(1 + sk - 2n)(1 + sk - 3n)\cdots[1 + sk - (s-1)n]$$

by

$$\left[\frac{1 + sk - n}{1 + sk - sn}\right]. \tag{14}$$

With these conventions series (13) becomes

$$\begin{aligned} y = y_0 &+ \frac{b}{nc}y_0^{1+k} + \frac{b^2}{2!\,n^2c^2}[1 + 2k - n]y_0^{1+2k} \\ &+ \frac{b^3}{3!\,n^3c^3}\left[\frac{1+3k-n}{1+3k-3n}\right]y_0^{1+3k} + \frac{b^4}{4!\,n^4c^4}\left[\frac{1+4k-n}{1+4k-4n}\right]y_0^{1+4k} + \cdots \\ &+ \frac{b^s}{s!\,n^sc^s}\left[\frac{1 + sk - n}{1 + sk - sn}\right]y_0^{1+sk} + \cdots. \end{aligned} \tag{15}$$

If series (15) is convergent, it will furnish a root of the three-term equation

$$ay^n + by^k + c = 0$$

for each one of the n values of y_0.

10. To test series (15) for convergency write the first n terms in regular order in a row, underneath this row the succeeding n terms and so on indefinitely. The terms of series (15) will now be arranged in n columns as follows:

$$
\begin{aligned}
(16)\qquad & y_0 \qquad + \frac{b}{nc} y_0^{1+k} + \cdots + \frac{b^{n-1}}{(n-1)!\, n^{n-1} c^{n-1}} \left[\frac{1+(n-1)k-n}{1+(n-1)k-(n-1)n} \right] y_0^{1+(n-1)k} \\
& + \frac{b^n}{n!\, n^n c^n} \left[\frac{1+nk-n}{1+nk-n^2} \right] y_0^{1+nk} + \frac{b^{n+1}}{(n+1)!\, n^{n+1} c^{n+1}} \left[\frac{1+(n+1)k-n}{1+(n+1)k-(n+1)n} \right] y_0^{1+(n+1)k} + \cdots + \frac{b^{2n-1}}{(2n-1)!\, n^{2n-1} c^{2n-1}} \left[\frac{1+(2n-1)k-n}{1+(2n-1)k-(2n-1)n} \right] y_0^{1+(2n-1)k} \\
& + \frac{b^{2n}}{(2n)!\, n^{2n} c^{2n}} \left[\frac{1+2nk-n}{1+2nk-2n^2} \right] y_0^{1+2nk} + \frac{b^{2n+1}}{(2n+1)!\, n^{2n+1} c^{2n+1}} \left[\frac{1+(2n+1)k-n}{1+(2n+1)k-(2n+1)n} \right] y_0^{1+(2n+1)k} + \cdots + \frac{b^{3n-1}}{(3n-1)!\, n^{3n-1} c^{3n-1}} \left[\frac{1+(3n-1)k-n}{1+(3n-1)k-(3n-1)n} \right] y_0^{1+(3n-1)k} \\
& \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& + \frac{b^{sn}}{(sn)!\, n^{sn} c^{sn}} \left[\frac{1+snk-n}{1+snk-sn^2} \right] y_0^{1+snk} + \frac{b^{sn+1}}{(sn+1)!\, n^{sn+1} c^{sn+1}} \left[\frac{1+(sn+1)k-n}{1+(sn+1)k-(sn+1)n} \right] y_0^{1+(sn+1)k} + \cdots + \frac{b^{sn-1}}{(sn-1)!\, n^{sn-1} c^{sn-1}} \left[\frac{1+(sn-1)k-n}{1+(sn-1)k-(sn-1)n} \right] y_0^{1+(sn-1)k} \\
& \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots
\end{aligned}
$$

This rearrangement of the terms of series (15) into the n columns of the table is permissible, inasmuch as throughout this investigation only absolute convergence is considered.

Cauchy's ratio test shows that each one of the n partial series composed of the terms in each of the n columns of the table is convergent when

$$\frac{b^n}{a^k c^{n-k}} < \frac{n^n}{k^k (n-k)^{n-k}}. \tag{17}$$

11. In like manner, if the algebraic functions defined by the equations

$$ay^n + by^k + cx = 0 \tag{2}$$

$$ay^n x + by^k + c = 0 \tag{3}$$

are expanded into power series in x by Lagrange's theorem, and if x is made unity in this power series, it is found that the resulting infinite series are convergent, provided

$$\frac{b^n}{a^k c^{n-k}} > \frac{n^n}{k^k (n-k)^{n-k}}. \tag{18}$$

12. If condition (18) is satisfied, equation (2) determines $n-k$ and equation (3) determines k roots of the three-term equation

$$ay^n + by^k + c = 0.$$

Either condition (17) or condition (18) must be satisfied, unless

$$\frac{b^n}{a^k c^{n-k}} = \frac{n^n}{k^k (n-k)^{n-k}}. \tag{19}$$

If condition (19) is satisfied, Raabe's test shows that the series obtained from equations (1), (2), (3) are all convergent.

13. The convergency conditions for equations (1), (2), (3) may be written by following these directions:

(a) To the left of the sign of inequality stands a fraction whose numerator contains the coefficient of the middle term of the three-term equation

$$ay^n + by^k + c = 0$$

and whose denominator contains the product of the coefficients of the end terms, the exponent of each coefficient being the difference of the exponents in the other two terms taken in order from left to right.

(b) The fraction to the right of the sign of inequality is obtained from the fraction to the left by replacing each coefficient by its exponent.

(c) The sign of inequality is $<$ when the term containing x is between the other two terms; if the term containing x is an end term the sign of inequality is $>$.

14. The following table exhibits the convergency conditions for the series obtained from equations (1), (2), (3) and the number of roots of the three-term equation

$$ay^n + by^k + c = 0$$

furnished by each one of these series.

$$(20)\quad \begin{array}{lll} (1)\ ay^n + by^kx + c = 0 & n & \dfrac{b^n}{a^kc^{n-k}} \leqq \dfrac{n^n}{k^k(n-k)^{n-k}}, \\ (2)\ ay^n + by^k + cx = 0 & n-k & \left\{ \dfrac{b^n}{a^kc^{n-k}} \geqq \dfrac{n^n}{k^k(n-k)^{n-k}}. \right. \\ (3)\ ay^nx + by^k + c = 0 & k & \end{array}$$

The roots of the three-term equation can always be expressed in infinite series.

III. The Four-term Equation.

15. In the four-term equation

$$ay^n + by^k + cy^l + d = 0$$

the two terms from which the factor x is to be omitted can be selected in six different ways. This gives rise to the six equations:

$$(21)\qquad ay^n + by^kx + cy^lx + d = 0$$

$$(22)\qquad ay^n + by^k + cy^lx + dx = 0$$

$$(23)\qquad ay^nx + by^k + cy^lx + d = 0$$

$$(24)\qquad ay^nx + by^k + cy^l + dx = 0$$

$$(25)\qquad ay^nx + by^kx + cy^l + d = 0$$

$$(26)\qquad ay^n + by^kx + cy^l + dx = 0$$

Each one of these six equations defines y as an algebraic function of x. The y of equation (21) may be expanded into a power series in x by any one of the three methods of articles 7, 8, 9. Using the symbol (14) and denoting $(-d/a)^{1/n}$ by y_0, this power series, when x is made unity, becomes

$$
\begin{aligned}
(27)\ y = {} & y_0 \\
& + \frac{b}{nd} y_0^{1+k} \qquad + \frac{c}{nd} y_0^{1+l} \\
& + \frac{b^2}{2!\, n^2 d^2}[1 + 2k - n]\, y_0^{1+2k} + \frac{2bc}{2!\, n^2 d^2}[1 + k + l - n]\, y_0^{1+k+l} + \frac{c^2}{2!\, n^2 d^2}[1 + 2l - n]\, y_0^{1+2} \\
& + \frac{b^3}{3!\, n^3 d^3}\left[\frac{1+3k-n}{1+3k-3n}\right] y_0^{1+3k} + \frac{3b^2c}{3!\, n^3 d^3}\left[\frac{1+2k+l-n}{1+2k+l-3n}\right] y_0^{1+2k+l} + \frac{3bc^2}{3!\, n^3 d^3}\left[\frac{1+k+2l-n}{1+k+2l-3n}\right] y_0^{1+k+2l} + \cdots \\
& + \frac{b^4}{4!\, n^4 d^4}\left[\frac{1+4k-n}{1+4k-4n}\right] y_0^{1+4k} + \frac{4b^3c}{4!\, n^4 d^4}\left[\frac{1+3k+l-n}{1+3k+l-4n}\right] y_0^{1+3k+l} + \frac{4b^2c^2}{4!\, n^4 d^4}\left[\frac{1+2k+2l-n}{1+2k+2l-4n}\right] y_0^{1+2k+2l} + \cdots \\
& + \frac{b^5}{5!\, n^5 d^5}\left[\frac{1+5k-n}{1+5k-5n}\right] y_0^{1+5k} + \frac{5b^4c}{5!\, n^5 d^5}\left[\frac{1+4k+l-n}{1+4k+l-5n}\right] y_0^{1+4k+l} + \frac{5b^3c^2}{5!\, n^5 d^5}\left[\frac{1+3k+2l-n}{1+3k+2l-5n}\right] y_0^{1+3k+2l} + \cdots \\
& \qquad \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

16. The infinite series composed of the terms of the left-hand column of the value of y is convergent when

$$(28) \qquad \frac{b^n}{a^k d^{n-k}} \leqq \frac{n^n}{k^k(n-k)^{n-k}},$$

and if condition (28) is satisfied this infinite series furnishes the solution of the three-term equation

$$(29) \qquad ay^n + by^k + d = 0.$$

It is found that each one of the infinite series composed of the terms of the respective columns of (27) is convergent when (28) is satisfied. It follows that (27) may be written

$$(29) \qquad y = X_0 + \frac{c}{nd} y_0^{\,l} X_1 + \frac{c^2}{n^2 d^2} y_0^{\,2l} X_2 + \frac{c^3}{n^3 d^3} y_0^{\,3l} X_3 + \cdots,$$

where X_0, X_1, X_2, X_3, $\cdots$, stand for the sums of convergent series. If now X is the largest of the numbers X_0, X_1, X_2, X_3, $\cdots$,

$$(30) \qquad y \gtreqless X\left(1 + \frac{c}{nd} y_0^{\,l} + \frac{c^2}{n^2 d^2} y_0^{\,2l} + \frac{c^3}{n^3 d^3} y_0^{\,3l} + \cdots\right),$$

and this last value of y is convergent when

$$(31) \qquad \frac{c}{nd} y_0^{\,l} < 1.$$

Affecting both sides of this inequality by the exponent n, this convergency condition may be written

$$(32) \qquad \frac{c^n}{a^l d^{n-l}} < n^n.$$

17. Conditions (28) and (32) are sufficient for the absolute convergence of (27). Condition (28) shows that the series which determines the roots of the three-term equation

$$(29) \qquad ay^n + by^k + d = 0$$

is found from

$$(33) \qquad ay^n + by^k x + d = 0.$$

The columns of (27) after the first are the corrections which must be applied to the roots of the three-term equation (29) to obtain the roots of the four-term equation

$$ay^n + by^k + cy^l + d = 0$$

18. If the two terms in the second row of (27) are interchanged and the consequent changes are made throughout (27), the left-hand column in the resulting value of y is convergent if

(34) $$\frac{c^n}{a^l d^{n-l}} \leqq \frac{n^n}{l^l(n-l)^{n-l}},$$

and the entire expression for y is convergent if in addition

(35) $$\frac{b^n}{a^k d^{n-k}} < n^n.$$

Conditions (34) and (35) are sufficient for the absolute convergence of the new series for y.

Condition (34) shows that the series which determines the solutions of the three-term equation

(36) $$ay^n + cy^l + d = 0$$

is found from

(37) $$ay^n + cy^l x + d = 0.$$

This series is the left-hand column of the value of y.

Condition (35) shows that the series of corrections which must be applied to the roots of the three-term equation (36) to obtain the solution of the four-term equation

$$ay^n + by^k + cy^l + d = 0$$

is convergent.

19. From equation (21) by omitting in succession each of the terms containing x are obtained the equations

(33) $$ay^n + by^k x + d = 0$$

(37) $$ay^n + cy^l x + d = 0$$

The convergency conditions (28) and (34) may be written from equations (33) and (37) respectively by following the directions (a), (b), (c) of article 13. The left-hand members of the conditions (32) and (35), together with the character of the signs of inequality, may be written from equations (37) and (33) respectively by following the same directions. The right-hand member of conditions (32) and (35) is formed by writing the difference of the exponents of the two terms of (21) which do not contain x and giving this difference an exponent equal to itself. It will be found that when the sign of inequality is $>$ in convergency conditions corresponding to conditions (32) and (35) the right-hand member is the reciprocal of what it is when the sign of inequality is $<$.

20. In like manner two sets of conditions sufficient for the absolute convergence of the infinite series giving the roots of the four-term equation obtained from each one of the equations (21), (22), (23), (24), (25), (26) may be written.

The convergency conditions for all these infinite series may be taken from the following table, in which the signs of equality of the limiting conditions of convergence have been omitted.

(38)	$\frac{b^n}{a^k d^{n-k}}$	$\frac{c^n}{a^l d^{n-l}}$	$\frac{b^{n-l}}{a^{k-l} c^{n-k}}$	$\frac{c^k}{b^l d^{k-l}}$	
(21) $ay^n+by^kx+cy^lx+d=0$	$<$	$<$			n^n
(22) $ay^n+by^k+cy^lx+dx=0$	$>$		$>$		$(n-k)^{n-k}$
(23) $ay^nx+by^k+cy^lx+d=0$	$>$			$<$	k^k
(24) $ay^nx+by^k+cy^l+dx=0$			$>$	$>$	$(k-l)^{k-l}$
(25) $ay^nx+by^kx+cy^l+d=0$		$>$		$>$	l^l
(26) $ay^n+by^kx+cy^l+dx=0$			$<$	$>$	$(n-l)^{n-l}$
	$\frac{n^n}{k^k(n-k)^{n-k}}$	$\frac{n^n}{l^l(n-l)^{n-l}}$	$\frac{(n-l)^{n-l}}{(k-l)^{k-l}(n-k)^{n-k}}$	$\frac{k^k}{l^l(k-l)^{k-l}}$	

In this table the signs of the two inequalities which constitute the convergency conditions of the series obtained from the equations (21) to (26) are placed to the right of the respective equations. The left-hand member of each inequality is at the top of the column in which the sign of inequality stands. The right-hand member of one inequality must be taken at the bottom of the column in which the sign of inequality stands; the right-hand member of the second inequality is the expression at the right of the row in which the sign of inequality stands when the sign of inequality is $<$, when the sign of inequality is $>$ the right-hand member of the inequality is the reciprocal of this expression.

21. The following table exhibits one set of convergency conditions of the infinite series which give the roots of the three-term equation

$$ay^n+by^k+d=0$$

together with the equations from which these series are derived and the number of the roots given by each series, and also the conditions sufficient for the absolute convergence of the series of corrections which must be applied to the roots of this three-term equation to obtain the roots of the four-term equation

$$ay^n + by^k + cy^l + d = 0.$$

(39)		$\frac{b^n}{a^k d^{n-k}}$	$\frac{c^n}{a^l d^{n-l}}$	$\frac{b^{n-l}}{a^{k-l} c^{n-k}}$	$\frac{c^k}{b^l d^{k-l}}$
I $ay^n + by^k x + d = 0$	n	$<$	$<$		
II $ay^n + by^k + dx = 0$	$n-k$	$>$		$>$	
II $ay^n x + by^k + d = 0$	k	$>$			$<$
		$\frac{n^n}{k^k (n-k)^{n-k}}$	n^n	$\frac{1}{(n-k)^{n-k}}$	k^k

22. The substitution

(40) $$y = z^s,$$

where s is a positive integer, transforms the four-term equation

$$ay^n + by^k + cy^l + d = 0$$

into the four-term equation

(41) $$az^{ns} + bz^{ks} + cz^{ls} + d = 0.$$

The table of convergency conditions for equation (41) corresponding to table (39) is

(42)		$\left(\frac{b^n}{a^k d^{n-k}}\right)^s$	$\left(\frac{c^n}{a^l d^{n-l}}\right)^s$	$\left(\frac{b^{n-l}}{a^{k-l} c^{n-k}}\right)^s$	$\left(\frac{c^k}{b^l d^{k-l}}\right)^s$
I $az^{ns} + bz^{ks} x + d = 0$	ns	$<$	$<$		
II $az^{ns} + bz^{ks} + dx = 0$	$ns-ks$	$>$		$>$	
II $az^{ns} x + bz^{ks} + d = 0$	ks	$>$			$<$
		$\left(\frac{n^n}{k^k (n-k)^{n-k}}\right)^s$	$(sn)^{sn}$	$\frac{1}{(sn-sk)^{sn-sk}}$	$(sk)^{sk}$

The three-term equations

$$ay^n + by^k + d = 0$$

$$az^{ns} + bz^{ks} + d = 0$$

for all values of s have the same convergency conditions.

If the inequality

$$\frac{c^n}{a^l d^{n-l}} < n^n$$

of table (39) is not satisfied, it is always possible to take s sufficiently large so that the corresponding inequality

$$\frac{c^n}{a^l d^{n-l}} < s^n n^n$$

of table (42) will be satisfied.

In like manner, if the inequalities

$$\frac{b^n}{a^k d^{n-k}} > \frac{n^n}{k^k(n-k)^{n-k}}, \quad \frac{b^{n-l}}{a^{k-l} c^{n-k}} > \frac{1}{(n-k)^{n-k}}, \quad \frac{c^k}{b^l d^{k-l}} < k^k$$

of table (39) are not satisfied simultaneously, it is always possible to take s sufficiently large so that the corresponding inequalities of table (42)

$$\frac{b^n}{a^k d^{n-k}} > \frac{n^n}{k^k(n-k)^{n-k}}, \quad \frac{b^{n-l}}{a^{k-l} c^{n-k}} > \frac{1}{s^{n-k}(n-k)^{n-k}}, \quad \frac{c^k}{b^l d^{k-l}} < s^k k^k$$

will be satisfied simultaneously.

To the convergency conditions of table (42) must be added the limiting convergency conditions obtained by replacing in the first column of inequality signs of table (42) each inequality sign by the equality sign.

It follows that it is always possible to determine s so that all the roots of the four-term equation

(41) $$az^{ns} + bz^{ks} + cz^{ls} + d = 0$$

may be derived from the roots of the three-term equation

(42) $$az^{ns} + bz^{ks} + d = 0.$$

The roots of the four-term equation

$$ay^n + by^k + cy^l + d = 0$$

are found from the roots of equation (41) by substituting in

(40) $$y = z^s.$$

23. While table (42) shows the possibility of expressing all the

roots of equation (41) in infinite series, the method of article (22) requires the determination of the ns roots of equation (41) to find the n roots of the four-term equation

$$ay^n + by^k + cy^l + d = 0.$$

This method is therefore to be avoided in practice when possible.

The following table exhibits the conditions sufficient for the absolute convergence of the infinite series which give the roots of the four-term equation obtained from the four groups of equations. The series obtained from each group of equations determine all the roots of the four-term equation. The convergency conditions must be taken from this table as in article 20, and the limiting convergency conditions must be taken into account.

A less inclusive set of conditions sufficient for the absolute convergence of the series which give the roots of the four-term equation derived from the groups of equations of table (43) is obtained by taking the second member of each inequality from the bottom of the column in which the sign of inequality stands.

(43)			$\frac{b^n}{a^k d^{n-k}}$	$\frac{c^n}{a^l d^{n-l}}$	$\frac{b^{n-l}}{a^{k-l} c^{n-k}}$	$\frac{c^k}{b^l d^{k-l}}$	
I	$ay^n+by^kx+cy^lx+d=0$	n	$<$	$<$			n^n
II	$ay^n+by^k+cy^lx+dx=0$	$n-k$	$>$		$>$		$(n-k)^{n-k}$
	$ay^nx+by^k+cy^lx+d=0$	k	$>$			$<$	k^k
III	$ay^n+by^kx+cy^l+dx=0$	$n-l$		$>$	$<$		$(n-l)^{n-l}$
	$ay^nx+by^kx+cy^l+d=0$	l		$>$		$>$	l^l
IV	$ay^n+by^k+cy^lx+dx=0$	$n-k$	$>$		$>$		$(n-k)^{n-k}$
	$ay^nx+by^k+cy^l+dx=0$	$k-l$			$>$	$>$	$(k-l)^{k-l}$
	$ay^nx+by^kx+cy^l+d=0$	l		$>$		$>$	l^l
			$\frac{n^n}{k^k(n-k)^{n-k}}$	$\frac{n^n}{l^l(n-l)^{n-l}}$	$\frac{(n-l)^{n-l}}{(k-l)^{k-l}(n-k)^{n-k}}$	$\frac{k^k}{l^l(k-l)^{k-l}}$	

It is only when the convergency conditions of the groups I, II, III, IV, together with the corresponding limiting convergency conditions fail simultaneously that the use of equation (41) becomes necessary.

IV. The Five-term Equation.

24. In the five-term equation

$$ay^n + by^k + cy^l + dy^m + l = 0$$

the two terms from which the factor x is to be omitted can be selected in ten different ways. This gives rise to the ten equations:

$$ay^n + by^kx + cy^lx + dy^mx + l = 0 \tag{44}$$

$$ay^n + by^k + cy^lx + dy^mx + lx = 0 \tag{45}$$

$$ay^n + by^kx + cy^l + dy^mx + lx = 0 \tag{46}$$

$$ay^n + by^kx + cy^lx + dy^m + lx = 0 \tag{47}$$

$$ay^nx + by^k + cy^l + dy^mx + lx = 0 \tag{48}$$

$$ay^nx + by^k + cy^lx + dy^m + lx = 0 \tag{49}$$

$$ay^nx + by^kx + cy^l + dy^m + lx = 0 \tag{50}$$

$$ay^nx + by^k + cy^lx + dy^mx + l = 0 \tag{51}$$

$$ay^nx + by^kx + cy^l + dy^mx + l = 0 \tag{52}$$

$$ay^nx + by^kx + cy^lx + dy^m + l = 0 \tag{53}$$

Each one of these ten equations defines y as an algebraic function of x which may be expanded into a power series by any one of the methods of articles 7, 8, 9.

25. The terms of the power series expressing the value of the algebraic function defined by equation (44), using the symbol (14) and placing $y_0 = (-l/a)^{1/n}$, when x is made unity, may be arranged as follows:

$$
\begin{array}{llll}
(54)\ y = & y_0 & & \\
 & +\dfrac{b}{en}\,y_0^{1+k} & +\dfrac{c}{en}\,y_0^{1+l} & \\
 & +\dfrac{b^2}{2\,!\,e^2n^2}\,[1+2k-n]\,y_0^{1+2k} & +\dfrac{2bc}{2\,!\,e^2n^2}\,[1+k+l-n]\,y_0^{1+k+l} & +\dfrac{c^2}{2\,!\,e^2n^2}\,[1+2l-n]\,y_0^{1+2l} \\
 & +\dfrac{b^3}{3\,!\,e^3n^3}\left[\dfrac{1+3k-n}{1+3k-3n}\right]y_0^{1+3k} & +\dfrac{3b^2c}{3\,!\,e^3n^3}\left[\dfrac{1+2k+l-n}{1+2k+l-3n}\right]y_0^{1+2k+l} & +\dfrac{3bc^2}{3\,!\,e^3n^3}\left[\dfrac{1+k+2l-n}{1+k+2l-2n}\right]y_0^{1+k+2l}+\cdots \\
 & \vdots & \vdots & \vdots \\
 & +\dfrac{d}{en}\,y_0^{1+m} & & \\
 & +\dfrac{2bd}{2\,!\,e^2n^2}\,[1+k+m-n]\,y_0^{1+k+m} & +\dfrac{2cd}{2\,!\,e^2n^2}\,[1+l+m-n]\,y_0^{1+l+m} & \\
 & +\dfrac{3b^2d}{3\,!\,e^3n^3}\left[\dfrac{1+2k+m-n}{1+2k+m-3n}\right]y_0^{1+2k+m} & +\dfrac{3bcd}{3\,!\,e^3n^3}\left[\dfrac{1+k+l+m-n}{1+k+l+m-3n}\right]y_0^{1+k+l+m} & +\dfrac{3c^2d}{3\,!\,e^3n^3}\left[\dfrac{1+2l+m-n}{1+2l+m-3n}\right]y_0^{1+2l+m}+\cdots \\
 & \vdots & \vdots & \vdots
\end{array}
$$

26. The first group of terms of (54) is the infinite series which gives the solution of the four-term equation

$$ay^n + by^k + cy^l + l = 0$$

obtained from the equation

$$ay^n + by^k x + cy^l x + l = 0$$

provided the conditions

$$\frac{b^n}{a^k e^{n-k}} < \frac{n^n}{k^k(n-k)^{n-k}}, \quad \frac{c^n}{a^l e^{n-l}} < n^n,$$

are satisfied.

The second group of terms has the common factor

$$\frac{d}{en} y_0^m,$$

and the successive groups of terms respectively the common factors

$$\frac{d^2}{e^2 n^2} y_0^{2m}, \quad \frac{d^3}{e^3 n^3} y_0^{3m}, \quad \frac{d^4}{e^4 n^4} y_0^{4m}, \quad \cdots.$$

The convergency conditions of the successive groups of terms are identical with the convergency conditions of the first group. It follows that (54) may be written

$$(55) \qquad y = Y_0 + Y_1 \frac{d}{en} y_0^m + Y_2 \frac{d^2}{e^2 n^2} y_0^{2m} + Y_3 \frac{d^3}{e^3 n^3} y_0^{3m} + \cdots,$$

where $Y_0, Y_1, Y_2, Y_3, Y_4, \cdots$, represent the sums of convergent infinite series.

If Y denotes the largest of the numbers $Y_0, Y_1, Y_2, Y_3, \cdots$,

$$(56) \qquad y \leqq Y\left(1 + \frac{d}{en} y_0^m + \frac{d^2}{e^2 n^2} y_0^{2m} + \frac{d^3}{e^3 n^3} y_0^{3m} + \cdots\right).$$

The series (56) is convergent provided

$$(57) \qquad \frac{d}{en} y_0^m < 1.$$

If both members of the inequality (57) are affected by the exponent n, condition (57) becomes

$$(58) \qquad \frac{d^n}{a^m e^{n-m}} < n^n.$$

The conditions sufficient for the absolute convergence of (54) are therefore

$$(59)\qquad \frac{b^n}{a^k e^{n-k}} < \frac{n^n}{k^k(n-k)^{n-k}}, \quad \frac{c^n}{a^l e^{n-l}} < n^n, \quad \frac{d^n}{a^m e^{n-m}} < n^n.$$

27. When the conditions (59) are satisfied the first group of terms of (54) gives the roots of the four-term equation

$$ay^n + by^k + cy^l + l = 0$$

expressed in the series obtained from the equation

$$ay^n + by^k x + cy^l x + l = 0$$

and the successive groups of (54) are the series of corrections which must be applied to the roots of this four-term equation to obtain the roots of the five-term equation

$$ay^n + by^k + cy^l + dy^m + l = 0.$$

28. If in the first row of (54) either of the terms

$$\frac{c}{en} y_0^{\,1+l}, \quad \frac{d}{en} y_0^{\,1+m}$$

is placed first and the consequent changes in (54) are made, the convergency conditions of the two new series are found to be

$$(60)\qquad \frac{b^n}{a^k e^{n-k}} < n^n, \quad \frac{c^n}{a^l e^{n-l}} < \frac{n^n}{l^l(n-l^{n-l}}, \quad \frac{d^n}{a^m e^{n-m}} < n^n;$$

$$(61)\qquad \frac{b^n}{a^k e^{n-k}} < n^n, \quad \frac{c^n}{a^l e^{n-l}} < n^n, \quad \frac{d^n}{a^m e^{n-m}} < \frac{n^n}{m^m(n-m)^{n-m}}.$$

In the limiting convergency conditions the signs of inequality in the first inequality of (59), in the second inequality of (60) and in the third inequality of (61) must be replaced by the equality sign.

The conditions sufficient for the absolute convergence of (54) may be written from equation (44) by the method stated in article 19.

In like manner the conditions sufficient for the absolute convergence of the series obtained from equations (45) to (53) may be written.

The convergency conditions for all these series may be taken from the following table. The convergency conditions are taken from the table by the method stated in article 20, except that the right-hand members of two inequalities must be determined from the expressions at the right of the row in which the sign of inequality stands.

(62)	$\frac{b^n}{a^k e^{n-k}}$	$\frac{c^n}{a^l e^{n-l}}$	$\frac{d^n}{a^m e^{n-m}}$	$\frac{b^{n-m}}{a^{k-m} d^{n-k}}$	$\frac{b^{n-l}}{a^{k-l} c^{n-k}}$	$\frac{c^k}{b^l e^{k-l}}$	$\frac{d^k}{b^m e^{k-m}}$	$\frac{c^{n-m}}{a^{l-m} d^{n-l}}$	$\frac{d^l}{c^m e^{l-m}}$	$\frac{c^{k-m}}{b^{l-m} d^{k-l}}$	
(44)	$<$	$<$	$<$								n^n
(45)	$>$			$>$	$>$						$(n-k)^{n-k}$
(46)		$>$			$<$			$>$			$(n-l)^{n-l}$
(47)			$>$	$<$				$<$			$(n-m)^{n-m}$
(48)					$>$	$>$				$>$	$(k-l)^{k-l}$
(49)				$>$			$>$			$<$	$(k-m)^{k-m}$
(50)								$>$	$>$	$>$	$(l-m)^{l-m}$
(51)	$>$					$<$	$<$				k^k
(52)		$>$				$>$			$<$		l^l
(53)			$>$				$>$		$>$		m^m
	$\frac{n^n}{k^k(n-k)^{n-k}}$	$\frac{n^n}{l^l(n-l)^{n-l}}$	$\frac{n^n}{m^m(n-m)^{n-m}}$	$\frac{(n-m)^{n-m}}{(k-m)^{k-m}(n-k)^{n-k}}$	$\frac{(n-l)^{n-l}}{(k-l)^{k-l}(n-k)^{n-k}}$	$\frac{k^k}{l^l(k-l)^{k-l}}$	$\frac{k^k}{m^m(k-m)^{k-m}}$	$\frac{(n-m)^{n-m}}{(l-m)^{l-m}(n-l)^{n-l}}$	$\frac{l^l}{m^m(l-m)^{l-m}}$	$\frac{(k-m)^{k-m}}{(l-m)^{l-m}(k-l)^{k-l}}$	

29. The following table exhibits one set of conditions sufficient for the absolute convergence of the infinite series which give the roots of the four-term equation

$$ay^n + by^k + cy^l + l = 0$$

together with the equations from which these series are obtained and the number of roots given by each series, and also the conditions sufficient for the absolute convergence of the series of corrections which must be applied to the roots of this four-term equation to obtain the roots of the five-term equation

$$ay^n + by^k + cy^l + dy^m + l = 0.$$

(63)			$\frac{b^n}{a^k e^{n-k}}$	$\frac{c^n}{a^l e^{n-l}}$	$\frac{b^{n-l}}{a^{k-l} e^{n-k}}$	$\frac{c^k}{b^l e^{k-l}}$	$\frac{d^n}{a^m e^{n-m}}$	$\frac{b^{n-m}}{a^{k-m} d^{n-k}}$	$\frac{d^k}{b^m e^{k-m}}$
I	$ay^n+by^kx+cy^lx+e=0$	n	$<$	$<$			$<$		
II	$ay^n+by^k+cy^lx+ex=0$	$n-k$	$>$		$>$			$>$	
	$ay^nx+by^k+cy^lx+e=0$	k	$>$			$<$			$<$
			$\frac{n^n}{k^k(n-k)^{n-k}}$	n^n	$\frac{1}{(n-k)^{n-k}}$	k^k	n^n	$\frac{1}{(n-k)^{n-k}}$	k^k

30. The substitution

(64) $$y = z^s,$$

where s is a positive integer, transforms the five-term equation

$$ay^n + by^k + cy^l + dy^m + l = 0$$

into another five-term equation

(65) $$ay^{ns} + by^{ks} + cy^{ls} + dy^{ms} + l = 0.$$

An examination of the table of convergency conditions for equation (65) corresponding to table (63), shows that it is always possible so to determine s that the convergency conditions for the series obtained from the equation

(66) $$ay^{ns} + by^{ks}x + cy^{ls}x + dy^{ms}x + l = 0$$

or from the pair of equations

(67) $$ay^{ns} + by^{ks} + cy^{ls}x + dy^{ms}x + lx = 0$$

(68) $$ay^{ns}x + by^{ks} + cy^{ls}x + dy^{ms}x + l = 0$$

are satisfied. Hence it is always possible to determine all the roots of a five-term equation by means of series.

31. The method of article 30 requires the determination of the ns roots of equation (65) in order to find the n roots of the five-term equation

$$ay^n + by^k + cy^l + dy^m + l = 0.$$

The use of this method becomes necessary only when the convergency conditions of the seven groups of equations of the following table, together with the corresponding limiting convergency conditions fail simultaneously.

The convergency conditions must be taken from this table as in article 20.

A less inclusive set of congruency conditions may be taken from this table as in article 23.

(69)

			$\frac{b^n}{a^k e^{n-k}}$	$\frac{c^n}{a^l e^{n-l}}$	$\frac{d^n}{a^m e^{n-m}}$	$\frac{b^{n-m}}{a^{k-m} d^{n-k}}$	$\frac{b^{n-l}}{a^{k-l} c^{n-k}}$	$\frac{c^k}{b^l e^{k-l}}$	$\frac{d^k}{b^m e^{k-m}}$	$\frac{c^{n-m}}{a^{l-m} d^{n-l}}$	$\frac{d^l}{c^m e^{l-m}}$	$\frac{c^{k-m}}{b^{l-m} d^{k-l}}$	
I	$ay^n + by^k + cy^lx + dy^mx + ex = 0$	n	$<$	$<$	$<$								n^n
II	$ay^n + by^k + cy^lx + dy^mx + ex = 0$	$n-k$	$>$			$>$	$>$						$(n-k)^{n-k}$
	$ay^nx + by^k + cy^lx + dy^mx + e = 0$	k	$>$					$<$	$<$				k^k
III	$ay^n + by^kx + cy^l + dy^mx + ex = 0$	$n-l$		$>$			$<$			$>$			$(n-l)^{n-l}$
	$ay^nx + by^kx + cy^l + dy^mx + e = 0$	l		$>$				$>$			$<$		l^l
IV	$ay^n + by^kx + cy^lx + dy^m + ex = 0$	$n-m$			$>$	$<$				$<$			$(n-m)^{n-m}$
	$ay^nx + by^kx + cy^lx + dy^m + e = 0$	m			$>$				$>$		$>$		m^m
V	$ay^n + by^k + cy^lx + dy^mx + ex = 0$	$n-k$	$>$			$>$	$>$						$(n-k)^{n-k}$
	$ay^nx + by^k + cy^l + dy^mx + ex = 0$	$k-l$					$>$	$>$				$>$	$(k-l)^{k-l}$
	$ay^nx + by^kx + cy^l + dy^mx + e = 0$	l		$>$			$>$			$<$			l^l
VI	$ay^n + by^kx + cy^l + dy^mx + ex = 0$	$n-l$		$>$			$<$			$>$			$(n-l)^{n-l}$
	$ay^nx + by^kx + cy^l + dy^m + ex = 0$	$l-m$								$>$	$>$	$>$	$(l-m)^{l-m}$
	$ay^nx + by^kx + cy^lx + dy^m + e = 0$	m		$>$				$>$		$>$			m^m
VII	$ay^n + by^k + cy^lx + dy^mx + ex = 0$	$n-k$	$>$			$>$	$>$						$(n-k)^{n-k}$
	$ay^nx + by^k + cy^l + dy^mx + ex = 0$	$k-l$					$>$	$>$				$>$	$(k-l)^{k-l}$
	$ay^nx + by^kx + cy^l + dy^m + ex = 0$	$l-m$								$>$	$>$	$>$	$(l-m)^{l-m}$
	$ay^nx + by^kx + cy^lx + dy^m + e = 0$	m		$>$				$>$		$>$			m^m
			$\frac{n^n}{k^k(n-k)^{n-k}}$	$\frac{n^n}{l^l(n-l)^{n-l}}$	$\frac{n^n}{m^m(n-m)^{n-m}}$	$\frac{(n-m)^{n-m}}{(k-m)^{k-m}(n-k)^{n-k}}$	$\frac{(n-l)^{n-l}}{(k-l)^{k-l}(n-k)^{n-k}}$	$\frac{k^k}{l^l(k-l)^{k-l}}$	$\frac{k^k}{m^m(k-m)^{k-m}}$	$\frac{(n-m)^{n-m}}{(l-m)^{l-m}(n-l)^{n-l}}$	$\frac{l^l}{m^m(l-m)^{l-m}}$	$\frac{(k-m)^{k-m}}{(l-m)^{l-m}(k-l)^{k-l}}$	

V. Conclusion.

32. In the algebraic equation of t terms

$$f(y) = 0$$

the two terms from which the factor x is to be omitted can be selected in

$$\frac{t(t-1)}{2}$$

ways. Each one of the resulting equations defines y as an algebraic function of x, and each algebraic function of x can be expanded into a power series in x by the methods used to obtain the corresponding expansions for the three-, four- and five-term equations. When x is made unity in these power series the resulting series become the roots of the t-term equation and a table of convergency conditions for these series analogous to tables (20), (38), (62) can be set up. In fact, this table may be written mechanically by following the directions of article 19.

33. If in the t-term equation the substitution

$$y = z^s$$

is made, a table of convergency conditions analogous to tables (39), (63) can be set up, and the value of s can be determined so that this table of conditions shows that it is possible to obtain all the roots of the transformed equation from the series derived either from the equation in which x is omitted from the first and last terms, or from the two equations in which x is omitted from the first and second, and from the second and last terms respectively. The roots of the given equation are then found from the roots of the transformed equation by substituting in

$$y = z^s.$$

34. Finally, tables of convergency conditions analogous to tables (43), (64) can be set up for the t-term equation, and it is necessary to use the transformed equation only when the convergency condi-

tions of all the groups of this table, together with the corresponding limiting convergency conditions, fail simultaneously.

35. It follows that all the roots of an algebraic equation of any number of terms, that is, of any algebraic equation, can be expressed in infinite series by the method of this investigation.

Lehigh University, Bethlehem, Pa.,
April 2, 1908.

Stated Meeting May 1, 1908.

Treasurer Jayne in the Chair.

Dr. Martin G. Brumbaugh, a newly elected member, was presented to the chair, and took his seat in the Society.

Letters were read, accepting election to membership from

Martin Grove Brumbaugh, Ph.D., Philadelphia.
Walter Bradford Cannon, A.M., M.D., Boston, Mass.
James Christie, Philadelphia.
Edward Washburn Hopkins, Ph.D., LL.D., New Haven, Conn.
Josiah Royce, Ph.D., LL.D., Cambridge, Mass.
Jacob G. Schurman, Ph.D., Ithaca, N. Y.
Edward Anthony Spitzka, M.D., Philadelphia.
Robert Williams Wood, Ph.D., Baltimore.

Mr. R. H. Mathews presented some "Notes on Australian Laws of Descent."

Professor Albert A. Michelson, of Chicago, was unanimously elected a Vice-President to fill the unexpired term of Professor George F. Barker, resigned.

Stated Meeting May 15, 1908.

Curator Doolittle in the Chair.

Letters were read accepting membership from

William Hallock, Ph.D., New York City.
Leonard Pearson, M.D., Philadelphia.
Charles Henry Smyth, Ph.D., Princeton, N. J.
John Robert Sitlington Sterrett, Ph.D., Ithaca, N. Y.
Ernest Nys, Brussels.

From Professor Albert A. Michelson accepting election to the Vice-Presidency to fill an unexpired term.

From the Committee of Organization of the Third Congrès International de Botanique, announcing that the Congress will be held at Brussels from May 14–22, 1910, and inviting the Society to be represented by delegates.

Dr. H. M. Chance read a paper on "The Origin of Bombshell Ore" (see page 135), which was discussed by Mr. Sanders, Mr. Jayne and Professor Doolittle.